Particular solutions to multidimensional PDEs with KdV-type nonlinearity.

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Abstract

We consider a class of particular solutions to the (2+1)-dimensional nonlinear partial differential equation (PDE) $u_t + \partial_{x_2}^n u_{x_1} - u_{x_1} u = 0$ (here n is any integer) reducing it to the ordinary differential equation (ODE). In a simplest case, $n = 1$, the ODE is solvable in terms of elementary functions. Next choice, $n = 2$, yields the cnoidal waves for the special case of Zakharov-Kuznetsov equation. The proposed method is based on the deformation of the characteristic of the equation $u_t - uu_{x_1} = 0$ and might also be useful in study the higher dimensional PDEs with arbitrary linear part and KdV-type nonlinearity (i.e. the nonlinear term is $u_{x_1} u$).

1 Introduction

Method of characteristics [8] is an effective tool for integrability of first order nonlinear PDEs in arbitrary dimensions. Its modification allowing one to integrate (1+1)-dimensional vector equations (the so called generalized hodograph method) was introduced by Tsarev [1] and developed in refs.[2, 3, 4]. A class of matrix equations integrable by the method of characteristics is proposed in [5].

In this paper we develop a method of construction of the particular solutions to a new class of multidimensional PDEs using a deformation of characteristics of already known integrable first-order equations. We will refer to the following first order (1+1)-dimensional partial differential equation (PDE)

$$u_t - u_{x_1} u = 0, \tag{1}$$

which is a simplest equation implicitly solvable by the method of characteristics. It describes the break of the wave profile. It is well known that its general solution is implicitly given by the non-differential equation

$$x + ut = F(u), \tag{2}$$

which is a characteristic of eq.(1). Here F is an arbitrary function which may be fixed by the initial condition. We show that the properly modified equation (2) allows one to partially integrate the higher-dimensional PDEs, in particular, the (2+1)-dimensional PDE

$$u_t + \partial_{x_2}^n u_{x_1} - u_{x_1} u = 0. \tag{3}$$

Note that the term "partially integrable PDEs" has two different meanings. First, the PDE is partially integrable if the available solution space is not full (see examples in ref.[6]).

Second, the PDE is partially integrable if its solution space is described in terms of the lower dimensional PDEs [7]. Our nonlinear PDEs are partially integrable in both senses. More precisely, we consider a family of solutions to the $(2+1)$-dimensional nonlinear PDE (3) which is described by a nonlinear ordinary differential equation (ODE) with independent variable x_2 where x_1 and t appear as parameters. This method may be viewed as a deformation of the method of characteristics [8].

If $n = 1$, then eq.(3) reads

$$u_t + u_{x_1 x_2} - u u_{x_1} = 0. \tag{4}$$

In this case we represent the explicite solution parametrized by one arbitrary function of single variable. If $n = 2$, then eq.(28) yields a special case of Zakharov-Kuznetsov equation [9, 10, 11]

$$u_t + u_{x_1 x_2 x_2} - u u_{x_1} = 0. \tag{5}$$

Explicite solutions are represented in terms of elliptic functions (in particular, in terms of the cnoidal waves) and are parametrized by two arbitrary functions of single variable.

General algorithm for $(2 + 1)$-dimensional PDE is represented in the next section, Sec.2, together with examples of explicite solutions. The asymptotic solutions at large t are derived as well. Higher dimensional generalization of the proposed algorithm is discussed in Sec.3. Conclusions are given in Sec. 4.

2 General form of (2+1)-dimensional PDE reducible to ODE

We proceed with the (2+1)-dimensional nonlinear PDE and prove the following theorem.

Theorem 1. Let function u be a solution of the following ODE (a deformation of eq.(2))

$$x_1 + ut = \frac{1}{2}u^2 - \partial^n_{x_2} u \tag{6}$$

(n is an arbitrary integer) in the domain

$$a_1 < x_1 < b_1, \quad a_2 \le x_2 \le b_2, \quad t > 0 \tag{7}$$

of the space of variables x_1, x_2 with some boundary conditions

$$B^i u \equiv \alpha_{1i} u|_{x_2=a_2} + \alpha_{2i} u|_{x_2=b_2} + \alpha_{3i} \partial^i_{x_2} u|_{x_2=a_2} + \alpha_{4i} \partial^i_{x_2} u|_{x_2=b_2} = \chi_i, \quad i = 1, \dots, n-1, \tag{8}$$

where functions χ_i are constrained by the linear PDEs

$$(\chi_i)_t = (\chi_i)_{x_1} t + \alpha_{1i} + \alpha_{2i}, \quad i = 1, \dots, n-1. \tag{9}$$

Then the function u is a solution of the nonlinear PDE

$$u_t + \partial^n_{x_2} u_{x_1} - u u_{x_1} = 0, \tag{10}$$

with boundary conditions (8,9).

Proof. To derive eq.(10) from eq.(6) we, first, rewrite eq.(6) in the form

$$x_1 + ut + w = 0, \tag{11}$$

where

$$w(u) = \partial^n_{x_2} u - \frac{1}{2}u^2. \tag{12}$$

For the further analysis, we shall mention that eq.(10) admits a representation in the form

$$u_t + w_{x_1}(u) = 0. \tag{13}$$

Moreover, w satisfies equation

$$w_t + \partial^n_{x_2} w_x - u w_{x_1} = 0, \tag{14}$$

which may be written as

$$(\partial^n_{x_2} - u)(u_t + w_{x_1}) = 0. \tag{15}$$

Next, applying the differential operators ∂_t, ∂_{x_1} and $L = \partial_{x_1}\partial^n_{x_2}$ to eq.(11) we obtain:

$$\begin{aligned} E_t &:= u + t u_t + w_t = 0, & (16)\\ E_x &:= 1 + t u_{x_1} + w_{x_1} = 0, & (17)\\ E_L &:= t\partial^n_{x_2} u_{x_1} + \partial^n_{x_2} w_{x_1} = 0. & (18) \end{aligned}$$

Now let us consider the combination

$$E_t - E_x u + E_L \tag{19}$$

which reads

$$t(u_t + \partial^n_{x_2} u_{x_1} - u u_{x_1}) + w_t + \partial^n_{x_2} w_{x_1} - u w_{x_1} = 0. \tag{20}$$

In virtue of eqs.(14) and (15) we have $w_t + \partial^n_{x_2} w_x - u w_{x_1} = (\partial^n_{x_2} - u)(u_t + w_{x_1})$, so that eq.(20) can be written in the following form

$$(t + \partial^n_{x_2} - u)\psi = 0, \tag{21}$$

where ψ is defined as follows:

$$\psi = u_t + \partial^n_{x_2} u_{x_1} - u u_{x_1}. \tag{22}$$

Eq.(21) is the ODE for the function ψ. However, to get original PDE (10), we need the trivial solution to eq.(21): $\psi \equiv 0$. For this purpose we impose the following zero conditions on the boundary of domain (7):

$$B^i\psi \equiv \alpha_{1i}\psi|_{x2=a_2} + \alpha_{2i}\psi|_{x2=b_2} + \alpha_{3i}\partial^i_{x_2}\psi|_{x2=a_2} + \alpha_{4i}\partial^i_{x_2}\psi|_{x2=b_2} = 0, \;\; i = 1, \dots, n-1. \tag{23}$$

Now, using the uniqueness theorem for the linear homogeneous PDE with zero boundary conditions we conclude that the only solution to the boundary-value problem (21,23) is zero, i.e. $\psi \equiv 0$, which is nothing but eq.(10).

We shall note that conditions (23) given on the boundary of domain (7) constrain the boundary conditions for PDE (10). We derive this constraint assuming that the boundary conditions for PDE (10) are introduced by the operators B^i in eq.(23) (i.e. we consider the

boundary conditions (8)). For this purpose we apply operators B^i to E_x given by eq.(17) and subtract the result from eq.(23) obtaining

$$B^i(u_t - u_{x_1}t - 1) = 0. \tag{24}$$

From another hand, applying the operator $\partial_t - t\partial_{x_1}$ to eq.(8) we obtain

$$B^i(u_t - u_{x_1}t) = (\chi_i)_t - t(\chi_i)_{x_1}. \tag{25}$$

Subtracting eq.(25) from eq.(24) we result in eq.(9) for the functions $\chi_i(x_1, t)$, $i = 1, \ldots, n-1$. $\square$

Below we list several properties of the solution u.

1. As shown in Theorem 1, the zero boundary conditions (23) for the function ψ effect the boundary conditions for the function u as a solution of the nonlinear PDE (6). Namely, functions χ_i in boundary conditions (8) are not arbitrary but satisfy system of linear eqs.(9). This system can be readily integrated yielding

$$\chi_i(x_1, t) = t(\alpha_{1i} + \alpha_{2i}) + A_i(\eta), \quad i = 1, \ldots, n-1 \tag{26}$$

$$\eta = t^2 + 2x_1, \tag{27}$$

   where A_i are arbitrary functions of single argument.

2. In general, the domain (7) of variables x_1 and x_2 might be either bounded or not bounded, i.e. a_1 and a_2 might tend to $-\infty$ while b_1 and b_2 might tend to $+\infty$.

3. If the function u is solution to both eqs.(10) and (6) then it satisfies the linear (1+1)-dimensional PDE

$$u_t = tu_{x_1} + 1. \tag{28}$$

   To derive eq.(28) we differentiate eq.(6) with respect to x_1 and subtract the result from eq.(10). Note that relation (28) on the boundary of domain (7) is forced by relations (9), as follows from the proof of Theorem 1, see eqs.(24,25).

4. In general, eq.(6) is a nonlinear ODE. However, it is remarkable that the asymptotic u^{as} of the bounded solution in the domain (7) of variables x_1 and x_2 with the bounded parameters a_1 and a_2 at large t is described by the linear ODE

$$\partial^n_{x_2} u^{as} = -u^{as}t, \tag{29}$$

   because, in this case, we neglect terms x_1 and $\frac{1}{2}u^2$ in eq.(6). In addition, we may neglect x_1 in definition of η (27) and write

$$\eta^{as} \equiv t^2. \tag{30}$$

   Analysis of eq.(29) depends on the particular n and on the boundary condition. We consider eq.(29) in Secs.2.1 and 2.2 for particular examples of nonlinear PDE (10) with $n = 1$ and $n = 2$.

2.1 Simplest case $n = 1$. Family of explicite solutions of nonlinear PDE

If $n = 1$, then nonlinear PDE (10) reads

$$u_t + u_{x_1 x_2} - u u_{x_1} = 0. \tag{31}$$

In turn, eq.(6) reads

$$u_{x_2} = \frac{1}{2}u^2 - x_1 - ut. \tag{32}$$

Boundary condition (8) is represented by a single equation (we take $a_2 = 0$ without loss of generality) which we take in the simplest form

$$u|_{x_2=0} = \chi_1(t, x_1). \tag{33}$$

Here χ_1 has the form given in eq.(26) with $\alpha_{11} = 1$ and $\alpha_{21} = 0$:

$$\chi_1(x_1, t) = t + A_1(\eta). \tag{34}$$

Eq.(32) is the first order ODE with separable variables. It may be readily represented in the form

$$u_{x_2} = \frac{1}{2}(u - u_+)(u - u_-), \quad u_\pm = t \pm \sqrt{\eta}, \tag{35}$$

and integrated. General solution of eq.(35) reads:

$$u(x_1, x_2, t) \;\; = \;\; t + \sqrt{\eta}\frac{1 + C(x_1, t)e^{x_2\sqrt{\eta}}}{1 - C(x_1, t)e^{x_2\sqrt{\eta}}}. \tag{36}$$

Boundary condition (33) in virtue of eq.(34) prescribes the dependence of C only on the variable η: $C(x_1, t) \equiv C(\eta)$. This condition reads:

$$A_1(\eta) = \sqrt{\eta}\frac{1 + C(\eta)e^{x_2\sqrt{\eta}}}{1 - C(\eta)e^{x_2\sqrt{\eta}}} \tag{37}$$

and may be considered as the definition of $A_1(\eta)$ in terms of the arbitrary function $C(\eta)$. Thus, eq.(36) represents a family of solutions of eq.(10) parametrized by one function of single variable:

$$u(x_1, x_2, t) \;\; = \;\; t + \sqrt{\eta}\frac{1 + C(\eta)e^{x_2\sqrt{\eta}}}{1 - C(\eta)e^{x_2\sqrt{\eta}}}. \tag{38}$$

2.1.1 Asymptotics of bounded solution

Consider the problem in the domain (7) of variables x_1, x_2 with the finite bounds a_1 and b_1 at large t. Then we shall use eq.(30) instead of (27) for η. Consequently, the asymptotic solution will not depend on x_1, which follows from eq.(38) for solution u. Let us expand solution (38) in powers of $e^{-x_2 t}$ and write its first nontrivial term as

$$u = C^{as}(t)e^{-x_2 t}, \quad C(t) = -\frac{2t}{C^{as}(t)}, \tag{39}$$

where $C^{as}(t)$ must be a bounded function of t. The same expression may be obtained directly from asymptotic equation (29) with $n = 1$. The boundary condition (33) in virtue of eq.(34) yields

$$A_1(t) = -t + C^{as}(t). \tag{40}$$

We see that asymptotic solution (39) vanishes at $t \to \infty$.

2.2 Case $n = 2$. Cnoidal waves for a particular case of Zakharov-Kuznetsov equation

If $n = 2$, then nonlinear PDE (10) reads

$$u_t + u_{x_1 x_2 x_2} - u u_{x_1} = 0, \tag{41}$$

which is a particular case of the Zakharov-Kuznetsov equation [9] when the transversal part of the two-dimensional Laplacian dominates (i.e., if the Debye radios in plasma is small in comparison with the $r_H = c_s/w_{Hi}$, where c_s is the sound velocity and w_{Hi} is the ion cyclotron frequency in plasma [9]). We choose the boundary conditions (8) in the form of two following equations

$$u|_{x2=a_2} = \chi_1(t, x_1), \quad u|_{x2=b_2} = \chi_2(t, x_1), \tag{42}$$

where χ_1, χ_2 are given by eqs.(26) with $\alpha_{11} = \alpha_{22} = 1$, $\alpha_{12} = \alpha_{21} = 0$:

$$\chi_i(t, x_1) = t + A_i(\eta), \quad i = 1, 2. \tag{43}$$

In our case, eq.(6) reads

$$u_{x_2 x_2} = \frac{1}{2}(u^2 - 2tu - 2x_1). \tag{44}$$

It is equivalent to the following first-order ODE:

$$u_{x_2} = \pm\frac{1}{\sqrt{3}}\sqrt{u^3 - 3tu^2 - 6x_1 u + \tilde{C}_1(x_1, t)}\,, \tag{45}$$

where $\tilde{C}_1$ is a function to be fixed by the boundary conditions (42). Introducing field v related with u by the equation

$$v = u - t, \tag{46}$$

we write eq.(45) as

$$v_{x_2} = \pm\frac{1}{2\sqrt{3}}\sqrt{4v^3 - 12(t^2 + 2x_1)v - 4C_1(x_1, t)}, \quad C_1(x_1, t) = 6tx_1 + 2t^3 - \tilde{C}_1(x_1, t). \tag{47}$$

The solution of eq.(47) is the elliptic Weierstrass function with invariants $g_2 = 12(t^2 + 2x_1)$ and $g_3(C_1(x_1, t)) = 4C_1(x_1, t)$:

$$v = u - t = \mathcal{W}\left(\pm\frac{x_2}{2\sqrt{3}} + C_2(x_1, t), g_2, g_3(C_1(x_1, t))\right). \tag{48}$$

The functions C_i, $i = 1, 2$ may be found from boundary conditions (42). Putting successively $x_2 = a_2$ and $x_2 = b_2$ in eq.(48) and using eq.(43) we see that C_i must depend only on η. Thus, eq.(48) represents solution of eq.(41) parametrized by two arbitrary functions of single variable:

$$u = t + \mathcal{W}\left(\pm\frac{x_2}{2\sqrt{3}} + C_2(\eta), g_2, g_3(C_1(\eta))\right). \tag{49}$$

Boundary conditions (42) in virtue of eqs.(43) read

$$\begin{aligned} A_1(\eta) &= \mathcal{W}\left(\pm\frac{a_2}{2\sqrt{3}}C_2(\eta), g_2, g_3(C_1(\eta))\right), \\ A_2(\eta) &= \mathcal{W}\left(\pm\frac{b_2}{2\sqrt{3}}C_2(\eta), g_2, g_3(C_1(\eta))\right). \end{aligned} \tag{50}$$

which must be taken as the definitions of the functions $A_i(\eta)$ in terms of the arbitrary functions $C_i(\eta)$, $i = 1, 2$.

Now we consider a particular case when all the roots of the polynomial under the square root in the rhs of eq.(47), i.e the roots of the polynomial

$$v^3 - 3(t^2 + 2x_1)v - C_1(x_1, t) = 0, \tag{51}$$

are real. These roots read

$$\begin{aligned} p_1 &= d + \frac{\eta}{d}, \\ p_2 &= \frac{1}{2}d(-1 + i\sqrt{3}) - \frac{1}{2d}(1 + i\sqrt{3})\eta = -\frac{p_1}{2} + \frac{\sqrt{3}i}{2}\left(d - \frac{\eta}{d}\right), \\ p_3 &= \frac{1}{2}d(-1 - i\sqrt{3}) - \frac{1}{2d}(1 - i\sqrt{3})\eta = -\frac{p_1}{2} - \frac{\sqrt{3}i}{2}\left(d - \frac{\eta}{d}\right) = -p_1 - p_2, \end{aligned} \tag{52}$$

where

$$d = 2^{-1/3}\left(C_1(x_1, t) + \sqrt{C_1^2(x_1, t) - 4\eta^3}\right)^{1/3}. \tag{53}$$

Thus, all p_i are parametrized by one function $C_1(x_1, t)$ which will be fixed by the boundary conditions. These roots are real if $|d|^2 = \eta$, which is the identity provided $C_1^2(x_1, t) - 4\eta^3 \le 0$. In this case, let

$$v_1(x_1, t) = \min(p_1, p_2, p_3), \quad v_3(x_1, t) = \max(p_1, p_2, p_3), \tag{54}$$

and $v_2(x_1, t)$ be the remaining root out of the list (p_1, p_2, p_3), i.e. $v_1 \le v_2 \le v_3$. Now we write eq.(47) as

$$v_{x_2} = \pm\frac{1}{\sqrt{3}}\sqrt{(v - v_1)(v - v_2)(v - v_3)}. \tag{55}$$

Let us assume that v is bounded as

$$v_1 \le v \le v_2. \tag{56}$$

Then the substitution

$$q=\sqrt{\frac{v-v_1}{v_2-v_1}},\ \ 0\le q\le 1 \tag{57}$$

transforms eq.(55) into the following form

$$q_{x_2}=\pm\frac{1}{2}\sqrt{\frac{v_3-v_1}{3}}\sqrt{(1-q^2)(1-k^2q^2)}, \tag{58}$$

$$k^2=\frac{v_2-v_1}{v_3-v_1}. \tag{59}$$

Solution to this equation is the Jacoby elliptic function

$$q=\sqrt{\frac{u-t-v_1}{v_2-v_1}}=\mathrm{sn}\Big(\pm\frac{x_2\sqrt{v_3-v_1}}{2\sqrt{3}}+\tilde{C}_2(x_1,t);k\Big). \tag{60}$$

Solving eq.(60) for u, we obtain

$$u=t+v_1(\eta)+(v_2(x_1,t)-v_1(x_1,t))\mathrm{sn}^2\Big(\pm\frac{x_2\sqrt{v_3(x_1,t)-v_1(x_1,t)}}{2\sqrt{3}}+\tilde{C}_2(x_1,t);k\Big). \tag{61}$$

Next, boundary conditions (62) suggest us to take C_1 and $\tilde{C}_2$ as functions of η, $C_1(x_1,t)\equiv C_1(\eta)$, $\tilde{C}_2(x_1,t)\equiv\tilde{C}_2(\eta)$. This means that $v_i(x_1,t)\equiv v_i(\eta)$, $i=1,2,3$. In addition, $\mathrm{sn}^2(x)$ is an even function of argument. Consequently, we may rewrite eq.(61) as (remember that $v_i(\eta)$, $i=1,2,3$, depend on $C_1(\eta)$ through eqs.(52,53))

$$u=t+v_1(\eta)+(v_2(\eta)-v_1(\eta))\mathrm{sn}^2\Big(\frac{x_2\sqrt{v_3(\eta)-v_1(\eta)}}{2\sqrt{3}}+C_2(\eta);k\Big),\ \ C_2(\eta)=\pm\tilde{C}_2(\eta). \tag{62}$$

Boundary conditions (42) in virtue of eqs.(43) yield

$$\begin{aligned} A_1(\eta)&=v_1(\eta)+(v_2(\eta)-v_1(\eta))\,\mathrm{sn}^2\Big(\frac{a_2\sqrt{v_3(\eta)-v_1(\eta)}}{2\sqrt{3}}+C_2(\eta);k\Big),\\ A_2(\eta)&=v_1(\eta)+(v_2(\eta)-v_1(\eta))\,\mathrm{sn}^2\Big(\frac{b_2\sqrt{v_3(\eta)-v_1(\eta)}}{2\sqrt{3}}+C_2(\eta);k\Big), \end{aligned} \tag{63}$$

which must be viewed as the definitions of A_i in terms of the arbitrary functions C_i, $i=1,2$. Consequently, eq.(62) represents a family of solutions of eq.(41) with two arbitrary functions C_i, $i=1,2$, of single variable η.

2.2.1 Asymptotic of bounded solutions

We consider the bounded solution on the domain (7) of variables x_1 and x_2 with finite bounds a_1 and b_1 at large t. Asymptotic eq.(29) reads:

$$u^{as}_{x_2x_2}=-tu^{as}. \tag{64}$$

This equation has the following general solution

$$u^{as}=C_1^{as}(x_1,t)\cos\Big(\sqrt{t}x_2+C_2^{as}(x_1,t)\Big), \tag{65}$$

where C_1^{as} is an amplitude and C_2^{as} is a phase. We consider $C_1^{as}(x_1,t) \geq 0$ without loss of generality (sign "$-$" may be embedded in C_2^{as}). At large t, we use eq.(30) instead of (27) for η. Then A_i, $i=1,2$, may be considered as functions of t. The boundary functions $\chi_i(t)$ (43) become bounded if

$$A_i(t) = -t + B_i(t), \quad i=1,2, \tag{66}$$

where $B_i(t)$, $i=1,2$, are bounded functions of argument. Boundary conditions (42) at $t\to\infty$ suggest us to take C_i^{as} as functions of t, so that eq.(65) reads

$$u^{as} = C_1^{as}(t)\cos\Big(\sqrt{t}x_2 + C_2^{as}(t)\Big). \tag{67}$$

Thus, asymptotic (67) does not depend on x_1 and is parametrized by two arbitrary functions of t, $C_i^{as}(t)$, $i=1,2$. In the asymptotic case, boundary conditions (42) in virtue of eqs.(43) read

$$B_1(t) = C_1^{as}(x_1,t)\cos\Big(a_2\sqrt{t} + C_2^{as}(t)\Big), \quad B_2(t) = C_1^{as}(x_1,t)\cos\Big(b_2\sqrt{t} + C_2^{as}(t)\Big). \tag{68}$$

which must be taken as definitions of $B_i(t)$ in terms of the arbitrary functions $C_i(t)$, $i=1,2$.

Asymptotics of cnoidal waves (62). Now we show that asymptotic solution (67) is nothing but the asymptotic of cnoidal wave (62).

Since η is given by eq.(30) at large t, we neglect x_1-dependence in C_i and take $C_1(t) = 2t^3 - t\hat{C}_1(t)$ with bounded $\hat{C}_1(t)\geq 0$. Then eq.(53) yields $d^{as} = t + \frac{i\sqrt{\hat{C}_1}}{3}$. Next we write the roots $p_i^{as} \equiv p_i|_{t\to\infty}$, given by eqs.(52), as follows:

$$p_1^{as} = 2t, \quad p_2^{as} = -t - \sqrt{\frac{\hat{C}_1}{3}}, \quad p_3^{as} = -t + \sqrt{\frac{\hat{C}_1}{3}}. \tag{69}$$

Thus, in the asymptotic limit, we have explicite map between v_i^{as} and p_i^{as},

$$v_1^{as} = p_2^{as} = -t - \sqrt{\frac{\hat{C}_1}{3}}, \quad v_2^{as} = p_3^{as} = -t + \sqrt{\frac{\hat{C}_1}{3}}, \quad v_3^{as} = p_1^{as} = 2t, \tag{70}$$

unlike the non-asymptotic case, see eqs.(54). Now we may calculate

$$v_2^{as} - v_1^{as} = 2\sqrt{\frac{\hat{C}_1}{3}}, \quad v_3^{as} - v_1^{as} = 3t + \sqrt{\frac{\hat{C}_1}{3}}. \tag{71}$$

Consequently, for k^2 in eq.(59) we have

$$k^2 = \frac{2\sqrt{\frac{\hat{C}_1}{3}}}{3t + \sqrt{\frac{\hat{C}_1}{3}}} \to 0, \quad \text{at} \quad t\to\infty. \tag{72}$$

Since $\operatorname{sn}(x;k)|_{k=0} = \sin(x)$, and $\sin^2 x = \frac{1}{2}(1-\cos 2x)$, we reduce eq.(62) to eq.(67) where

$$C_1^{as}(t) = -\sqrt{\frac{\hat{C}_1}{3}}, \quad C_2^{as}(t) = 2C_2(t). \tag{73}$$

In turn, conditions (63) result in eqs.(68) provided eqs.(66).

3 Generalization of algorithm to higher dimensions

The algorithm developed in Sec.2 may be generalized to higher dimension. In this section we introduce a large manifold of $(M+1)$-dimensional nonlinear PDEs with particular solutions satisfying either M- or $(M-1)$-dimensional nonlinear PDEs where the independent variables t (and sometimes x_1) appears as parameter. The increase in the dimensionality of the original nonlinear PDE is achieved through the increase in the dimensionality of its linear part, while the nonlinearity remains the same.

We shall remark that one could apply this algorithm to larger class of evolution PDEs with integro-differential linear part. This would be also important because this structure has, in particular, the multidimensional dispersion-less KP. In this respect, we have to remember ref.[12], where another type of particular solutions to this equation were studied and the wave breaking for such solutions was explained. However, multidimensional evolution PDEs with integro-differential linear part are out of the scope of this paper.

We formulate the following theorem.

Theorem 2. Let the differential operator L have the structure $L = \partial_{x_1}\tilde{L}$, where $\tilde{L}(\partial_{x_i},\ i = 1, 2\dots, M)$ is an arbitrary M-dimensional linear differential operator. Let u be a solution of the M-dimensional PDE

$$x_1 + ut = \frac{1}{2}u^2 - \tilde{L}u \tag{74}$$

in the M-dimensional domain of the space of the variables x_i, $i = 1, \dots, M$,

$$a_i \le x_i \le b_i, \quad i = 1, \dots, M, \quad t > 0 \tag{75}$$

with the complete set of boundary conditions

$$B^i u = \chi_i, \quad i = 1, \dots, K, \tag{76}$$

where K is some integer and functions χ_i, $i = 1, \dots, K$, satisfy constraints

$$(\chi_i)_t - (\chi_i)_{x_1} = B^i 1, \quad i = 1, \dots, K. \tag{77}$$

Then the function u is a solution of the $(M+1)$-dimensional PDE

$$u_t + Lu - uu_{x_1} = 0. \tag{78}$$

Proof. To derive eq.(78) from eq.(74) we, first, write eq.(74) as follows:

$$x_1 + ut + w = 0, \tag{79}$$

where

$$w = \tilde{L}u - \frac{1}{2}u^2. \tag{80}$$

Similar to eq.(10), eq.(78) admits a representation in the form

$$u_t + w_{x_1} = 0, \tag{81}$$

which is the M-dimensional PDE. Again, w satisfies the equation

$$w_t + Lw - uw_{x_1} = 0, \tag{82}$$

which may be written as

$$(\tilde{L} - u)(u_t + w_{x_1}) = 0. \tag{83}$$

Next, we apply operators ∂_t, ∂_{x_1} and L to eq.(79) obtaining:

$$\begin{aligned} E_t &:= u + tu_t + w_t = 0, &(84)\\ E_x &:= 1 + tu_{x_1} + w_{x_1} = 0, &(85)\\ E_L &:= tLu + Lw = 0. &(86)\end{aligned}$$

Now we consider the following combination

$$E_t - E_x u + E_L \tag{87}$$

which reads

$$t(u_t + Lu - uu_{x_1}) + w_t + Lw - uw_{x_1} = 0. \tag{88}$$

In virtue of eqs.(82) and (83) we may write $w_t + Lw - uw_{x_1} = (\tilde{L} - u)(u_t + w_{x_1})$. Then eq.(88) gets the following form,

$$(t + \tilde{L} - u)\psi = 0, \tag{89}$$

where ψ is defined as follows:

$$\psi = u_t + Lu - uu_{x_1}. \tag{90}$$

Eq.(89) is the M-dimensional PDE for the function ψ. Let us impose the zero boundary conditions for the function ψ

$$B^i \psi = 0, \quad i = 1, 2, \dots, K. \tag{91}$$

Then, in virtue of the uniqueness theorem for the linear homogeneous PDEs, we conclude that $\psi \equiv 0$, which coincides with PDE (78).

Boundary conditions (91) lead to constraints (77) for the boundary functions χ_i, $i = 1, \dots, K$. These constraints may be derived similarly to the derivation of eqs.(9). First, we apply operators B^i to eq.(85):

$$B^i E_x = 0, \quad i = 1, 2, \dots, K. \tag{92}$$

Subtracting this equation from the eq.(91) we obtain

$$B^{(i)}(u_t - tu_{x_1} - 1) = 0, \quad i = 1, 2, \dots, K. \tag{93}$$

From another hand, applying the operator $\partial_t - t\partial_{x_1}$ to eq.(76) we obtain

$$B^i(u_t - u_{x_1} t) = (\chi_i)_t - t(\chi_i)_{x_1}, \quad i = 1, 2, \dots, K. \tag{94}$$

Finally, subtracting eq.(94) from eq.(93) we derive eq.(77) for the functions $\chi_i(x_1, t)$, $i = 1, \dots, K$. $\Box$

Function u possesses the properties similar to those of the function u in Sec.2.

1. In general, eq.(74) is the M-dimensional PDE. However, if the operator $\tilde{L} = \tilde{L}(\partial_{x_2}, \dots, \partial_M)$ (i.e. $\tilde{L}$ is the $(M-1)$-dimensional operator), then eq.(74) becomes the $(M-1)$-dimensional PDE. Therewith the variable x_1 appears as a parameter in this equation.

2. The zero boundary conditions (91) for the function ψ effect the boundary conditions (76) for the function u through the liner system of (1+1)-dimensional PDEs (77) for the functions χ_i, $i = 1, \dots, K$.

3. In general, the domain (75) of variables x_i, $i = 1, \dots, M$, might be either bounded or not bounded, i.e. a_i might tend to $-\infty$ while b_i might tend to ∞.

4. If the function u is a solution to both eqs.(78) and (74) then it satisfies the linear (1+1)-dimensional PDE (28). To derive this equation we differentiate eq.(74) with respect to x_1 and subtract the result from eq.(78). Note that this property on the boundary of domain (75) is forced by relations (77) as follows from the proof of Theorem 2, see eqs.(93,94).

5. In general, eq.(74) is a nonlinear ODE. However, it is remarkable that the asymptotic u^{as} of the bounded solution in the domain (75) of variables x_i, $i = 1, \dots, M$, with bounded parameters a_1 and b_1 at large t is described by the linear PDE

$$\tilde{L}u^{as} = -u^{as}t, \tag{95}$$

   because, in this case, we neglect terms x_1 and $\frac{1}{2}u^2$ in eq.(74). We also must use expression (30) instead of (27) for η.

   Analysis of this equation depends on the particular $\tilde{L}$ and will not be considered in this paper.

4 Conclusions

We consider a method of construction of the particular solutions to nonlinear PDE (10). In the particular cases, $n = 1, 2$, these solutions are expressed in terms of the either elementary functions with one arbitrary function of single variable ($n = 1$) or elliptic functions with two arbitrary functions of single variable ($n = 2$). The Zakharov-Kuznetsov equation describing the ion-sound waves in a low-pressure magnetized plasma is the physically applicable example of such systems [9]. We construct the cnoidal waves for the special case of the Zakharov-Kuzntesov equation, when the two-dimensional Laplacian becomes the one-dimensional transversal one. Asymptotics of these waves in the bounded space-domain are described by a linear PDE yielding oscillating behavior.

Generalization of this method to higher dimensions is briefly discussed. In general, a nonlinear $(M+1)$-dimensional PDE from the considered class (78) possesses the family of solutions satisfying either M- or $(M-1)$-dimensional differential equation (74). Although we consider only evolutionary type differential equations, a certain type of the evolutionary equations with integro-differential linear part might be also studied in this way.

Emphasize that we use deformation of characteristic (2) in our constructions. However, an open problem is whether deformations of characteristics of other multidimensional PDEs might be used in a similar way.

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